Calculus II - Day 10

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Taylor Series

Goals for today:

- Express commonly used transcendental functions like e^x and $\sin(x)$ as power series by using derivatives.
- Find and prove an infinite sum formula for e and π.

Announcements:

- MyLab 9 due Wednesday, no MyLab due Friday.
- Problem Set 5 due Friday, 10/18 (but you should do it before the midterm).
- Fill out the Qualtrics survey by tomorrow night.

Recall: We can represent the function $\frac{1}{1-x}$ as a power series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for all} \quad x \in (-1, 1)$$

How can we do this for other functions?
Definition: Suppose f is a function that has derivatives of all orders on an interval around some number a. The Taylor series of f centered at a is:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, this is called a Maclaurin series.
Ex. Let $f(x) = e^x$. Find the Maclaurin series.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$\frac{f^{(k)}(0)}{k!}$
0	e^x	1	$\frac{1}{0!} = 1$
1	e^x	1	$\frac{1}{1!} = 1$
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	e^x	1	$\frac{1}{k!}$

So we have:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Where does this converge?
Use the Ratio Test to find the interval of convergence:

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x}{k+1} \right| = |x| \lim_{k\to\infty} \frac{1}{k+1} = 0 \quad \text{for all}\, x$$

The interval of convergence is $(-\infty, \infty)$, so for all x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Plug in $x = 1$:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$$

Ex. Find the Maclaurin series for $\sin(x)$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$\frac{f^{(k)}(0)}{k!}$
0	$\sin(x)$	0	0
1	$\cos(x)$	1	1
2	$-\sin(x)$	0	0
3	$-\cos(x)$	-1	$\frac{-1}{3!} = \frac{-1}{6}$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$\frac{1}{5!} = \frac{1}{120}$

So we have:

$$0 + 1x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5 + \cdots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x) \quad \text{for all } x$$

We could find the Maclaurin series of $\cos(x)$ using the definition... or we could differentiate the series:

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)\cdot x^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Check interval of convergence:

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+2}}{(2k+2)!}\cdot\frac{2k!}{(-1)^k x^{2k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{x^2}{(2k+2)(2k+1)}\right| = x^2 \lim_{k\to\infty}\left|\frac{1}{(2k+2)(2k+1)}\right| = 0 \quad \text{for every } x$$

Therefore:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

When you differentiate or integrate a Taylor series, the radius of convergence (RoC) always stays the same, but the endpoints might change.

Ex. Find the Maclaurin series for $\cos(3x^2)$.

Make a substitution of $3x^2$ into the Maclaurin series for $\cos(x)$:

$$\cos(3x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (3x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k}(x^2)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 9^k x^{4k}}{(2k)!}$$

Ex. Find the Taylor series for $\ln(x)$ centered at $a = 1$.

k	$f^{(k)}(x)$	$f^{(k)}(1)$	$\frac{f^{(k)}(1)}{k!}$
0	$\ln(x)$	0	0
1	$\frac{1}{x}$	1	1
2	$-\frac{1}{x^2}$	-1	$\frac{-1}{2!} = \frac{-1}{2}$
3	$\frac{2}{x^3}$	2	$\frac{2}{3!} = \frac{1}{3}$
4	$-\frac{6}{x^4}$	-6	$\frac{-6}{4!} = \frac{-1}{4}$
5	$\frac{24}{x^5}$	24	$\frac{24}{5!} = \frac{1}{5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$\frac{(-1)^{k+1}\cdot(k-1)!}{x^k}$	$(-1)^{k+1}$	$\frac{(-1)^{k+1}}{k}$

So we have:

$$0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k} = \ln(x) \quad \text{on } (0, 2]$$

Similarly:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \quad \text{on } (-1, 1]$$

Proof of Interval of Convergence (IoC):

$$r = \lim_{k\to\infty} \left| \frac{(-1)^{k+2}x^{k+1}}{k+1} \cdot \frac{k}{(-1)^{k+1}x^k} \right| = \lim_{k\to\infty} \left| \frac{x \cdot k}{k+1} \right| = |x| \lim_{k\to\infty} \frac{k}{k+1} = |x| < 1$$

Therefore, the series converges on $(-1, 1)$.

Check endpoints:

For $x = 1$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}1^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

This series converges by the Alternating Series Test (AST).

For $x = -1$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} = \sum_{k=1}^{\infty} \frac{-1}{k} = -\sum_{k=1}^{\infty} \frac{1}{k}$$

This series diverges by the p-test.

Therefore, the interval of convergence is:

$$(-1, 1]$$

Ex. A formula for π:
Start with:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

For $\frac{1}{1+x^2}$, we get:

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

Term-by-term integration:

$$\int \frac{1}{1+x^2}\, dx = \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\, dx$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + C$$

Setting $\arctan(0) = 0 + C = 0$, we find $C = 0$, so:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad \text{for } x \in [-1, 1]$$

For $x = 1$:

$$\arctan(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Therefore:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Big Six Maclaurin Series: Put these on your cheat sheet!

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for } x \in (-\infty, \infty)$
- $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{for } x \in (-\infty, \infty)$
- $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for } x \in (-\infty, \infty)$
- $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \quad \text{for } |x| \le 1,\ x \ne -1$
- $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad \text{for } |x| \le 1$